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A Note on Fixed-Point Theorems

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In an earlier note, S. P. Singh gave an extension of a theorem of Brosowski in a normed linear space setting. Variants of this theorem are considered in the context of strictly convex, reflexive, and inner product spaces.

Let X be a normed linear space. $T: X \to X$ is *contractive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in X$. The set of *best C-approximants* to x, $P_C(x)$, consists of all y in C such that $\|y - x\| = \inf\{\|z - x\|: z \in C\}$. A subset C is *starshaped* if there is a point p in C such that $x \in C$ and $0 \leqslant \lambda \leqslant 1$ imply $\lambda p + (1 - \lambda)x \in C$. Every convex set is starshaped, but not conversely. X is *strictly convex* if $x, y \in X$, $x \neq y$, and $\|x\| = \|y\|$, imply $\|\frac{1}{2}(x + y)\| < \|x\|$.

Let T be a contractive operator on a normed linear space X. Let C be a subset of X, and x be a T-invariant point. Brosowski [1] proved the following:

THEOREM. *There is a y in $P_C(x)$, which is also T-invariant, provided*

(A) $T: C \to C$;

(B) $P_C(x)$ *is nonempty and compact; and*

(C) $P_C(x)$ *is convex.*

Singh [5] proved that the conclusion holds even when $P_C(x)$ is only starshaped. So we weaken (C) to $P_C(x)$ is starshaped. Singh remarks that T must be linear in Brosowski's theorem, but an examination of the proof shows that linearity is not used.

LEMMA. *If $T: \partial C \to C$, then $T: P_C(x) \to P_C(x)$.*

Proof. Let $y \in P_C(x)$. Every neighborhood of y contains a point strictly between x and y on the line segment $\{\lambda x + (1 - \lambda) y: 0 < \lambda < 1\}$. Such a point is closer to x than y is, so it cannot be in C (y is a *best* C-approximant to x). Thus y is not in the interior of C, and since $T: \partial C \to C$, Ty must be in

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C. $\|Ty - x\| = \|Ty - Tx\| \leqslant \|y - x\|$. Thus Ty must also be a best C-approximant to x.

Remarks. In Singh's proof, the only use made of (A) $T: C \to C$, is to prove that $T: P_C(x) \to P_C(x)$. So we weaken (A) to $T: \partial C \to C$. The theorem is still true and Singh's proof is still valid. Since there are several fixed-point theorems for operators satisfying $T: \partial C \to C$, the lemma may have independent interest. It is sufficient that $T: B \to B$ and B contains all line segments joining x and points in C. In this case, and in other theorems as well, ∂C can be taken to mean the boundary of C relative to B, in $T: \partial C \to C$.

Condition (B) $P_C(x)$ is nonempty and compact, may be difficult to verify in specific instances. This leads to the consideration of special cases when it is possible to replace compactness by weak compactness or prove $P_C(x)$ is nonempty. The following is a variant of a theorem due to Browder [2, Theorem 4]. By applying it with $D = P_C(x)$, it would be possible, for Hilbert spaces, to replace (B) by $P_C(x)$ is nonempty and weakly compact.

THEOREM 1. *Let D be a nonempty, weakly compact, and starshaped subset of a Hilbert space X. Suppose T is a contractive mapping of D into D. Then T has at least one fixed point.*

Proof. In this setting weakly compact is equivalent to weakly sequentially compact which is equivalent to weakly closed and norm bounded. Using these facts, note that Browder's proof holds if the fixed element v_0 of D, in his proof, is chosen to be the p in the definition of starshaped.

EXAMPLE. For convex sets, weakly closed and closed are equivalent. This is not true for even bounded starshaped sets. We given an example of a set in l_2 which is closed, bounded, and starshaped, but not weakly closed. Let $\{e_n\}_{n=0}^{\infty}$ be a countable orthonormal basis for l_2 and let

$$S = \{\lambda(e_0 + e_n): 0 \leqslant \lambda \leqslant 1, n > 0\}.$$

If $x \in S$, $\|x\| \leqslant \sqrt{2}$ and $\lambda 0 + (1 - \lambda) x = (1 - \lambda) x \in S$ for all $0 \leqslant \lambda \leqslant 1$. Thus S is bounded and starshaped.

S is not weakly closed. $(e_0 + e_k, e_j) \to (e_0, e_j)$ for all j. Since $\{e_j\}$ spans $l_2 \approx l_2^*$, $(e_0 - e_k, f) \to (e_0, f)$ for all $f \in l_2$. Thus $e_0 \notin S$ but $e_0 + e_k \to e_0$ weakly.

S is closed. Suppose $\{x_n\} \subset S$ and $x_n \to x$. We show that $x \in S$. For all k, $(x_n, e_k) \to (x, e_k)$.

Case A. There exists $k > 0$ such that $(x, e_k) \neq 0$. Then there is an N such that for all $n \geqslant N$, $(x_n, e_k) \neq 0$. $x_n = \lambda_n(e_0 + e_{k_n})$ and $(x_n, e_k) \neq 0$ implies $x_n = \lambda_n(e_0 + e_k)$ for all $n \geqslant N$. Thus eventually $\{x_n\}$ is in the line segment $[0, e_0 + e_k]$, which is closed, so $x \in S$.

Case B. For all $k > 0$, $(x, e_k) = 0$. Then $x = \sum_{j=0}^{\infty} \alpha_j e_j = \alpha_0 e_0$. Let $x_n = \lambda_n(e_0 + e_{k_n})$. Then $(x_n, e_0) \to (x, e_0)$ or $\lambda_n[(e_0, e_0) + (e_{k_n}, e_0)] \to \alpha_0(e_0, e_0)$. Thus $\lambda_n \to \alpha_0$. $\|x_n\| \to \|x\|$ or $\lambda_n \|e_0 + e_{k_n}\| \to |\alpha_0| \|e_0\|$ or $\sqrt{2}\,\lambda_n \to |\alpha_0|$. But $\sqrt{2}\,\lambda_n \to \sqrt{2}\,\alpha_0$. Thus $\alpha_0 = 0$ and $x = 0 \in S$.

Naturally, if C is not closed, $P_C(x)$ may be empty. If C is closed, so is $P_C(x)$, and $P_C(x)$ is always bounded. Furthermore, $P_C(x)$ is nonempty and compact, whenever C is *boundedly compact*, that is, the intersection of C with every closed bounded set is compact. This includes the cases of X being finite-dimensional and of C being compact. Of course, in infinite-dimensional spaces, it may be difficult to prove that C is compact (or even boundedly compact). This appears to limit the applications of the theorem. The only example Brosowski [1] gives is finite-dimensional. The facts above are all elementary.

The most direct way to prove that $P_C(x)$ is starshaped, is to know that C is convex. In this case, $P_C(x)$ is even convex and Brosowski's original theorem is sufficient. We now consider the consequences of assuming that C is convex. If C is closed and convex, $P_C(x)$ is also, and thus is weakly closed. It is wellknown [4, p. 258], that if X is reflexive, then weakly closed, norm bounded sets are weakly compact.

Even if C is not boundedly compact, $P_C(x)$ is nonempty provided that C is closed and convex, and X is reflexive [4, p. 277]. Clearly, if C is not closed, $P_C(x)$ may be empty. An example similar to the one above ($S = \{\lambda(e_0 + ((n+1)/n)\, e_n)\colon 0 \leqslant \lambda \leqslant 1, n > 0\}$) shows that in l_2 there is a closed, bounded, and starshaped set which contains no best approximants to a point (e_0). Thus convex cannot be replaced by starshaped here. Likewise, if X is not reflexive, there are even hyperplanes that contain no best approximants. We need the following result, which is elementary for R^2 and R^3.

LEMMA. *If* x^* *is a nonzero continuous linear functional on* X *and* $S = \{x \in X\colon x^*(x) = \|x^*\|\}$ *then* $\inf\{\|x\|\colon x \in S\} = 1$.

Proof. Let $x \in S$. $x^*(x) = \|x^*\|$. But $|x^*(x)| \leqslant \|x^*\| \|x\|$. So $1 \leqslant \|x\|$. Hence $\inf\{\|x\|\colon x \in S\} \geqslant 1$. Now $\|x^*\| = \sup\{|x^*(x)|/\|x\|\colon x \neq 0\}$. Let $a > 1$. There is some x such that $\|x^*\| \leqslant a\,|x^*(x)|/\|x\|$, i.e., $\|x^*\| \|x\|/|x^*(x)| \leqslant a$. Let $y = (\|x^*\|/x^*(x))\,x$. Since $x^*(y) = \|x^*\|$, $y \in S$. $\|y\| = \|x^*\| \|x\|/|x^*(x)| \leqslant a$. Thus $\inf\{\|x\|\colon x \in S\} \leqslant a$ for all $a > 1$.

THEOREM 3. *The following are equivalent:*

(A) *X is reflexive.*

(B) *For every closed convex set S, there is an $x_0 \in S$ such that* $\|x_0\| = \inf\{\|x\|\colon x \in S\}$.

(C) *For every closed, bounded, and convex set S, there is an $x_0 \in S$ such that $\|x_0\| = \inf\{\|x\| : x \in S\}$.*

(D) *For every hyperplane S of the form $\{x : x^*(x) = \|x^*\|\}$ for some x^*, there is an $x_0 \in S$ such that $\|x_0\| = \inf\{\|x\| : x \in S\}$.*

(E) *For every continuous linear functional x^*, there is an x_0 such that $x^*(x_0) = \|x^*\|$ and $\|x_0\| = 1$.*

Proof. Statement (A) $\Rightarrow$ (B) is standard [4, p. 277]. Statement (B) $\Rightarrow$ (C) is evident. Statement (C) $\Rightarrow$ (D). Let S be as in (D). S is convex and weakly closed, hence closed. If $S' = S \cap \{x : \|x\| \leqslant 2\}$ then S' is closed, bounded, and convex. Assuming (C), S' has a point of minimum norm, x_0. Since $\inf\{\|x\| : x \in S\} = 1$, S' has the same minimum norm as S. Thus $x_0 \in S$ and $\|x_0\| = 1 = \inf\{\|x\| : x \in S\}$. Statement (D) $\Rightarrow$ (E). Given x^*, let $S = \{x : x^*(x) = \|x^*\|\}$. By (D), there is an $x_0 \in S$ (i.e., $x^*(x_0) = \|x^*\|$) such that $\|x_0\| = \inf\{\|x\| : x \in S\}$. But this infimum is 1. Statement (E) $\Rightarrow$ (A). The hard part is due to James [3].

If X is strictly convex, $P_C(x)$ satisfies a rather restrictive condition. Every nontrivial convex combination of points from $P_C(x)$ is in the complement of C. We will use the following special case.

LEMMA. *Let X be a strictly convex normed linear space, $x \in X$, and C a subset of X. If p_1 and p_2 are best C-approximants to x, then $\frac{1}{2}(p_1 + p_2) \notin C$.*

Proof. Let $p = \frac{1}{2}(p_1 + p_2)$. $\|p - x\| = \|\frac{1}{2}(p_1 - x) + \frac{1}{2}(p_2 - x)\|$. Now $\|p_1 - x\| = \|p_2 - x\| = \inf\{\|y - x\| : y \in C\} = d$, and $\|(p_1 - x) - (p_2 - x) = \|p_1 - p_2\| > 0$. So by strict convexity, $\|p - x\| = \|\frac{1}{2}(p_1 - x) + \frac{1}{2}(p_2 - x)\| < \max\{\|p_1 - x\|, \|p_2 - x\|\}$. Thus $\|p - x\| < \inf\{\|y - x\| : y \in C\}$ and $p \notin C$.

THEOREM 3. *Let X be strictly convex and $P_C(x)$ nonempty and starshaped. Then $P_C(x) = \{x_0\}$ with $Tx_0 = x_0$.*

Proof. If $q \neq p$ are two points in $P_C(x)$, then $\frac{1}{2}p + \frac{1}{2}q \notin C$; hence $\frac{1}{2}p + \frac{1}{2}q \notin P_C(x)$. So if $P_C(x)$ is starshaped, it must be a singleton. Since $T: \partial C \to C$, $T: P_C(x) \to P_c(x)$. Thus $Tx_0 \in P_c(x) = \{x_0\}$.

Remarks. This shows that the theorem is needed only in spaces that are not srictly convex. Recall that Hilbert spaces, L_p-spaces, for $p > 1$, and uniformly convex spaces are all strictly convex (or rotund), and also reflexive. Yet the condition $P_C(x) \neq \varnothing$ does not hold in general unless C is convex and X is reflexive, (or C is boundedly compact). This seems to limit most applications to finite-dimensional spaces with the l_1 or l_∞ norms. These spaces are not strictly convex, but are reflexive. In finite-dimensional spaces, if C is closed and convex, conditions (B) and (C) hold. The following theorem may have practical applications.

THEOREM 4. *Let X be a strictly convex, reflexive space. Suppose $T: C \to X$, $T: \partial C \to C$, and C is closed and convex. If T can be extended to a contractive operator on the whole of X in such a way that there is a T-invariant point x in X, then there is a T-invariant point in C.*

Proof. $P_C(x) \neq \varnothing$ since X is reflexive. $P_C(x) = \{x_0\} = \{Tx_0\}$ since X is strictly convex. But $x_0 \in C$.

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